

General Theorems on Decoherence in the Thermodynamic Limit*

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Abstract

We extend the results on decoherence in the thermodynamic limit [M. Frasca, Phys. Lett. A **283**, 271 (2001)] to general Hamiltonians. It is shown that N independent particles, initially properly prepared, have a set of observables behaving classically in the thermodynamic limit. This particular set of observables is then coupled to a quantum system that in this way decoheres so to have the density matrix in a mixed form. This gives a proof of the generality of this effect.

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Decoherence [1] appears as a rather ubiquitous effect in quantum systems. Due to the interaction with a large environment, a quantum system tends to modify its quantum evolution from unitary, that means coherent, to a decaying form. In this terms, we can say that decoherence, as commonly understood, is a dissipative effect. Then, the density matrix containing interference terms, after a trace procedure on the environment degrees of freedom gets a mixed form so to have probabilities attributed to the outcome of possible measurements on the system. As a matter of fact, this does not solve the measurement problem in quantum mechanics [2]. The main question is that, after a measurement, one gets a pure state for the quantum system and not just the mixed form of the density matrix that is obtained by decoherence.

Anyhow, the appearance of decaying effects on unitary evolution of a quantum system is generally seen in experiments and one can safely affirm that decoherence is a rather well verified phenomenon [3]. In this paper we want to go one step beyond on the basis of a recent proposal for non dissipative decoherence in the thermodynamic limit [4]. We will prove that a pure state is indeed obtained when a large number of quantum systems interacts with another one, washing out superposition states and approaching in this way a possible solution to the measurement problem, for a single event, self consistently inside quantum mechanics. The essential characteristic of our approach is that unitary evolution is preserved and decoherence is dynamically produced.

First of all, we will prove a general theorem on the appearance of classical states in the dynamical evolution of an ensemble of non interacting quantum systems. We show that:

Theorem 1 (Classicality) *An ensemble of N non interacting quantum systems, for properly chosen initial states, has a set of operators $\{A_i, B_i, \dots : i = 1 \dots N\}$ from which one can derive a set of observables behaving classically.*

To prove this theorem we consider a Hamiltonian of the form

$$H = \sum_{i=1}^N H_i \quad (1)$$

and a set of observables $\{A_i, B_i, \dots : i = 1 \dots N\}$. For each system we take a set of orthonormal eigenfunctions $|\phi_i^k\rangle$, being $\langle\phi_i^k|\phi_i^l\rangle = \delta_{kl}$, in

a such a way to have the initial state

$$|\psi(0)\rangle = \prod_{i=1}^N |\phi_i^{k_i}\rangle \quad (2)$$

not being an eigenstate of the Hamiltonian (1). Firstly, we prove that the Hamiltonians H_i belong to the set of operators $\{A_i, B_i, \dots : i = 1 \dots N\}$, being the corresponding observable the Hamiltonian H . This means that the mean value of H on the state (2) is overwhelming large with respect to its quantum fluctuations in the thermodynamic limit. This proves that the set $\{A_i, B_i, \dots : i = 1 \dots N\}$ is not empty. Indeed, by straightforward algebra one has

$$\langle H \rangle = N\bar{H} \quad (3)$$

having set

$$\bar{H} = \frac{1}{N} \sum_{i=1}^N \langle \phi_i^{k_i} | H_i | \phi_i^{k_i} \rangle \quad (4)$$

that is an average of the mean values of each H_i . This proves that the mean value of H is proportional to N . Without much more difficulty we get

$$\langle H^2 \rangle = N\overline{H^2} + \sum_{i \neq j} \langle \psi(0) | H_i H_j | \psi(0) \rangle \quad (5)$$

being

$$\overline{H^2} = \frac{1}{N} \sum_{i=1}^N \langle \phi_i^{k_i} | H_i^2 | \phi_i^{k_i} \rangle \quad (6)$$

again an average of the mean values of the square of each H_i . We note at this stage that the last term in eq.(5) gives a contribution also proportional to N^2 . This gives at last the fluctuation

$$(\Delta H)^2 = \langle H^2 \rangle - \langle H \rangle^2 = N\overline{(\Delta H)^2}, \quad (7)$$

the mean value of H now removes the N^2 term, with

$$\overline{(\Delta H)^2} = \frac{1}{N} \sum_{i=1}^N [\langle \phi_i^{k_i} | H_i^2 | \phi_i^{k_i} \rangle - \langle \phi_i^{k_i} | H_i | \phi_i^{k_i} \rangle^2] \quad (8)$$

showing that the square of the fluctuation of H is proportional to the average of the fluctuations of each H_i . So, one has that the mean value of H is proportional to N while the fluctuation is proportional

to \sqrt{N} and then the former is overwhelming large with respect to the latter, if the initial Hamiltonian (1) has a large ensemble of systems composing it (thermodynamic limit). Being the quantum fluctuations negligible, one can say that the quantum system we are considering, in the thermodynamic limit, behaves classically with respect to the observable H , if properly prepared in the state $|\psi(0)\rangle$. So, the ensemble $\{A_i, B_i, \dots : i = 1 \dots N\}$ is not empty.

Now, we extend the proof to any other operator that can belong to the set $\{A_i, B_i, \dots : i = 1 \dots N\}$. If, for a given i , A_i commutes with H , it does not evolve in time, being a conserved observable, and the above argument for H applies straightforwardly. Instead, if, generally, $[A_i, H] \neq 0$ we have to study the time evolution of these observables by the Heisenberg equations of motion. By analogy with H we introduce the observable $A = \sum_{i=1}^N A_i$ (the same can be done with any other set of operators belonging to the given set), then (here and in the following $\hbar = 1$)

$$A(t) = e^{iHt} A e^{-iHt}. \quad (9)$$

With this definition it is not difficult to obtain, with the same state initial state,

$$\langle A(t) \rangle = N \overline{A(t)} \quad (10)$$

being

$$\overline{A(t)} = \frac{1}{N} \sum_{k=1}^N \langle \phi_k | e^{iH_k t} A_k e^{-iH_k t} | \phi_k \rangle, \quad (11)$$

and

$$[\Delta A(t)]^2 = \langle A(t)^2 \rangle - \langle A(t) \rangle^2 = N \overline{\Delta A(t)^2} \quad (12)$$

being

$$\overline{\Delta A(t)^2} = \frac{1}{N} \sum_{k=1}^N [\langle \phi_k | e^{iH_k t} A_k^2 e^{-iH_k t} | \phi_k \rangle - \langle \phi_k | e^{iH_k t} A_k e^{-iH_k t} | \phi_k \rangle^2], \quad (13)$$

proving finally the theorem as we have the fluctuation proportional to \sqrt{N} and the mean value proportional to N . We can recognize from this result a strict similarity with statistical mechanics as it should be expected from the start (see [5]).

Once we have such a set of observables, one may ask if these operators can indeed produce decoherence. This is the content of the next theorem:

Theorem 2 (Decoherence) *An ensemble of N non interacting quantum systems, having a set of operators $\{A_i, B_i, \dots : i = 1 \dots N\}$, from which one can derive a set of observables behaving classically, and strongly interacting with a quantum system through a Hamiltonian having forms like $V_0 \otimes \sum_{i=1}^N A_i$, can produce decoherence if properly initially prepared.*

By “strongly interacting” we mean that the Hamiltonian of the N non interacting quantum systems can be neglected, and perturbation theory can be applied. We want to use a theorem for strong coupling proved in Ref.[6]. In fact, the Hamiltonian of this system can be written, choosing as a observable $\sum_{i=1}^N A_i$ acting in the Hilbert space of the bath,

$$H_{SB} = H_S + \sum_{i=1}^N H_i + V_0 \otimes \sum_{i=1}^N A_i \quad (14)$$

being H_S the Hamiltonian of the quantum system, and V_0 an operator, acting in the Hilbert space of the system, coupling the quantum system to the bath of N non interacting systems. So, if we assume the coupling between the system and the bath to be very large, we can apply the theorem of Ref.[6] to the Hamiltonian in the interaction picture in the system’s variables

$$H_I = e^{iH_S t} V_0 e^{-iH_S t} \sum_{i=1}^N A_i, \quad (15)$$

stating that the strong coupling approximation is given by

$$|\psi(t)\rangle \approx \sum_n e^{i\dot{\gamma}_n t} e^{-iN\bar{a}v_n t} |v_n\rangle \langle v_n | \psi_S(0)\rangle \prod_{i=1}^N |\chi_i\rangle \quad (16)$$

being $|\psi_S(0)\rangle$ the initial state of the quantum system,

$$V_0 |v_n\rangle = v_n |v_n\rangle, \quad (17)$$

assuming a discrete spectrum and

$$\dot{\gamma}_n = \langle v_n | H_S | v_n \rangle. \quad (18)$$

The initial state of the bath is chosen in such a way to have $A_i |\chi_i\rangle = a_i |\chi_i\rangle$ and being $\bar{a} = \sum_{i=1}^N a_i / N$ a constant.

The state (16) has a quite interesting aspect as the phases of the oscillating exponentials, already at very small energy, can have a time

scale of the order of the Planck time in the thermodynamic limit, making senseless the possibility to observe such oscillations on the corresponding probabilities. This means, mathematically, that such probability oscillations are averaged away [7]. Indeed, for the density matrix of the system one has

$$\begin{aligned} \rho_S(t) &= \sum_n |\langle v_n | \psi_S(0) \rangle|^2 |v_n\rangle \langle v_n| \\ &+ \sum_{m \neq n} e^{i[\gamma_m - \gamma_n]t} e^{-iN\bar{a}[v_m - v_n]t} \langle v_m | \psi_S(0) \rangle \langle \psi_S(0) | v_n \rangle |v_m\rangle \langle v_n| \end{aligned} \quad (19)$$

with the interference terms being averaged away on a maximum time scale $\tau_M = 1/(N\bar{a} \min[v_m - v_n])$ being $\min[v_m - v_n]$ the minimal energy difference between the eigenvalues of V_0 . This time is really small already at energies of order of eV for N becoming very large and generally comparable with the Planck time.

Finally, we prove a general theorem in measure theory in quantum mechanics, stating that

Theorem 3 (Measure) *If the operator V_0 , strongly coupled to a quantum system with an observable of the ensemble of N non interacting systems, is linear in the generators of coherent states, Schrödinger cat states are washed out in the leading order of the coupling in the thermodynamic limit.*

We assume, initially, that the system has the following Hamiltonian, neglecting the bath contribution at the leading order in the strong coupling expansion,

$$H_F = \omega a^\dagger a + (\gamma a^\dagger + \gamma^* a) \otimes \sum_{i=1}^N A_i \quad (20)$$

so that, we have V_0 given by a linear combination of generators of coherent states [8]. Besides, the initial state is taken to be a superposition state as

$$|\psi(0)\rangle = \mathcal{N}(|\alpha e^{i\phi}\rangle + |\alpha e^{-i\phi}\rangle) \prod_{i=1}^N |\chi_i\rangle \quad (21)$$

being \mathcal{N} a normalization factor and $|\alpha e^{\pm i\phi}\rangle$ coherent states as to have a Schrödinger cat state [9]. At the leading order, we can write the unitary evolution operator as [10],

$$U_F(t) = e^{i\xi(t)} e^{-i\omega a^\dagger a t} \exp[\hat{\beta}(t)a^\dagger - \hat{\beta}(t)^* a] \quad (22)$$

being

$$\hat{\xi}(t) = \frac{\left(\sum_{i=1}^N A_i\right)^2 |\gamma|^2}{\omega^2} (\omega t - \sin(\omega t)) \quad (23)$$

and

$$\hat{\beta}(t) = \frac{\left(\sum_{i=1}^N A_i\right) \gamma}{\omega} (1 - e^{i\omega t}). \quad (24)$$

So, it is straightforward to obtain the wave function as

$$\begin{aligned} |\psi(t)\rangle \approx U_F(t) |\psi(0)\rangle &= e^{i\xi(t)} \mathcal{N} \left(e^{i\phi_1(t)} |\beta(t) e^{-i\omega t} + \alpha e^{i\phi - i\omega t}\rangle \right. \\ &+ \left. e^{i\phi_2(t)} |\beta(t) e^{-i\omega t} + \alpha e^{-i\phi - i\omega t}\rangle \right) \prod_{i=1}^N |\chi_i\rangle \end{aligned} \quad (25)$$

being now

$$\xi(t) = \frac{N^2 |\gamma|^2}{\omega^2} (\omega t - \sin(\omega t)) \quad (26)$$

and

$$\beta(t) = \frac{N\gamma}{\omega} (1 - e^{i\omega t}). \quad (27)$$

and

$$\phi_1(t) = -i \frac{\alpha}{2} [\beta(t) e^{-i\phi} - \beta^*(t) e^{i\phi}], \quad (28)$$

$$\phi_2(t) = -i \frac{\alpha}{2} [\beta(t) e^{i\phi} - \beta^*(t) e^{-i\phi}]. \quad (29)$$

The state (26), in the thermodynamic limit $N \rightarrow \infty$, reduces to a pure coherent state for the system, $|\beta(t) e^{-i\omega t}\rangle$, proving the main assertion of the theorem. The Schrödinger cat state is washed away in the thermodynamic limit.

To keep this letter shorter, we omit to prove that the system of the last theorem undergoes true decoherence. This can be seen from the interference term of the Wigner function that, in the thermodynamic limit, displays very rapid oscillations on a time scale of the Planck time or smaller, being not physical. We can apply the theory of divergent series [11] to assume in the sense of Abel or Euler that $\lim_{N \rightarrow \infty} \cos(Nf(t)) = 0$ and $\lim_{N \rightarrow \infty} \sin(Nf(t)) = 0$ and all boils down to an average in time. So, no interference term can be actually observed and true classical behavior emerges. This argument is similar to the one applied in the proof of the second theorem.

In conclusion, we have generalized our approach to the study of quantum mechanics in the thermodynamic limit, given in [4], to a large class of quantum systems, proving its wide applicability.

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